Resistance vs. Load Reliability Analysis

Let \( L \) be the load acting on a system (e.g. footing) and let \( R \) be the resistance (e.g. soil).

Then we are interested in controlling \( R \) such that the probability that \( R > L \) (i.e. the reliability) is acceptably high or, equivalently, that \( R < L \) (i.e. the failure probability) is acceptably low, where

\[
P[R < L] = \int_{-\infty}^{\infty} \int_{-\infty}^{l} f_{RL}(r,l) \, dr \, dl
\]

and where \( f_{RL}(r,l) \) is the joint (bivariate) distribution of \( R \) and \( L \).

\[
f_{RL}(r,l) \, dr \, dl = P[r < R \leq r + dr \cap l < L \leq l + dl]
\]
Bivariate Distributions

\[ P[l_1 < L \leq l_2 \cap r_1 < R \leq r_2] = \int_{l_1}^{l_2} \int_{r_1}^{r_2} f_{RL}(r,l) \, dr \, dl \]
Resistance vs. Load Reliability Analysis

The estimation of $f_{RL}(r,l)$ typically requires vast amounts of data, which is generally impractical.

Simplifications:
1. assume $R$ and $L$ are independent so that

$$f_{RL}(r,l) = f_R(r)f_L(l)$$

and

$$P[R < L] = \int_{-\infty}^{\infty} \int_{r<l} f_R(r)f_L(l) \, dr \, dl = \int_{-\infty}^{\infty} f_L(l) \int_{r<l} f_R(r) \, dr \, dl$$
Resistance vs. Load Reliability Analysis

2. Assume $R$ and $L$ are either normally or lognormally distributed. The event $\{R < L\}$ is the same as the events
   i) $\{R - L < 0\}$
   ii) $\{R/L < 1\}$

If both $R$ and $L$ are normally distributed, then

$$X = R - L$$

is also normally distributed with parameters

$$\mu_X = \mu_R - \mu_L$$

$$\sigma^2_X = \sigma^2_R + \sigma^2_L$$

(assuming $R$ and $L$ are independent).
Reliability Index

- the reliability index, $\beta$, is the number of standard deviations the mean is from failure.
- superior to the Factor-of-Safety approach because it depends on both the mean and the standard deviation.
- failure occurs if $X < 0$ (normal) or $\ln X < 0$ (lognormal).

Defining
\[ \beta = \frac{\mu_X}{\sigma_X}, \quad \text{(normal)} \]
\[ \beta = \frac{\mu_{\ln X}}{\sigma_{\ln X}}, \quad \text{(lognormal)} \]

$P[\text{failure}] = P[X < 0] = \Phi\left(-\frac{\mu_X}{\sigma_X}\right) = \Phi(-\beta), \quad \text{(normal)}$

$P[\text{failure}] = P[\ln X < 0] = \Phi\left(-\frac{\mu_{\ln X}}{\sigma_{\ln X}}\right) = \Phi(-\beta), \quad \text{(lognormal)}$
Resistance vs. Load Reliability Analysis

• Suppose load is normally distributed with mean 10 and standard deviation 3.
• Suppose resistance is normally distributed with mean 20 and standard deviation 4.
• Then $X = R - L$ has mean and variance:
  \[
  \mu_X = \mu_R - \mu_L = 20 - 10 = 10
  \]
  \[
  \sigma^2_X = \sigma^2_R + \sigma^2_L = 4^2 + 3^2 = 25
  \]
• Mean FS = $\mu_R / \mu_L = 20/10 = 2$
• Reliability index = $\beta = \mu_X / \sigma_X = 10/5 = 2$
• Probability of failure = $P[R < L] = P[X < 0] = \Phi\left(\frac{0 - 10}{5}\right) = \Phi(-2) = 0.023$
Resistance vs. Load Reliability Analysis

Now \[ P[R < L] = P[R - L < 0] = P[X < 0] = \Phi \left(-\frac{\mu_X}{\sigma_X}\right) \]

where \( \Phi(x) \) is the standard normal cumulative distribution function.
Resistance vs. Load Reliability Analysis

Alternatively, if $R$ and $L$ are lognormally distributed, then

$$X = \frac{R}{L}$$

is also lognormally distributed with parameters

$$\mu_{\ln X} = \mu_{\ln R} - \mu_{\ln L}$$

$$\sigma^2_{\ln X} = \sigma^2_{\ln R} + \sigma^2_{\ln L} \quad \text{(assuming independence)}$$

so that

$$P[R < L] = P[R/L < 1] = P[X < 1] = P[\ln X < 0]$$

$$= \Phi\left(-\frac{\mu_{\ln X}}{\sigma_{\ln X}}\right)$$
Resistance vs. Load Reliability Analysis

• Suppose load is lognormally distributed with mean 10 and standard deviation 3
• Suppose resistance is lognormally distributed with mean 20 and standard deviation 4.
• Then $X = R/L$ is lognormally distributed

\[
\begin{align*}
\mu_R &= 20, \quad \sigma_R = 4 \quad \rightarrow \\
\mu_{\ln R} &= \ln(\mu_R) - 0.5\sigma^2_{\ln R} = 2.976 \\
\sigma^2_{\ln R} &= \ln\left(1 + \frac{\sigma^2_R}{\mu^2_R}\right) = 0.0392 \\
\mu_L &= 10, \quad \sigma_L = 3 \quad \rightarrow \\
\mu_{\ln L} &= \ln(\mu_L) - 0.5\sigma^2_{\ln L} = 2.259 \\
\sigma^2_{\ln L} &= \ln\left(1 + \frac{\sigma^2_L}{\mu^2_L}\right) = 0.0862
\end{align*}
\]
Resistance vs. Load Reliability Analysis

\[ X = \frac{R}{L} \quad \rightarrow \quad \ln X = \ln R - \ln L \]

\[ \mu_{\ln X} = \mu_{\ln R} - \mu_{\ln L} = 0.717 \]

\[ \sigma^2_{\ln X} = \sigma^2_{\ln R} + \sigma^2_{\ln L} = 0.125 \quad \rightarrow \quad \sigma_{\ln X} = 0.354 \]

\[ \beta = \frac{\mu_{\ln X}}{\sigma_{\ln X}} = \frac{0.717}{0.354} = 2.02 \quad \rightarrow \quad P\left[ \frac{R}{L} < 1 \right] = \Phi(-2.02) = 0.022 \]
Reliability Index

$FS = 2$

$X = R - L$

$\beta = 2.00$

$P[R < L] = 0.023$

$\beta \sigma_X$

$X = r - l$
Reliability Index

More generally, system failure can be defined in terms of a failure or limit state function. Also called the safety margin

\[ M = g(Z_1, Z_2, ...) \]

Failure occurs when \( M = g(Z_1, Z_2, ...) < 0 \). In this case, the reliability index is defined as

\[ \beta = \frac{\mu_M}{\sigma_M} \]

Problem: different choices of the function \( M \) lead to different reliability indices (e.g. \( M = R - L \) or \( M = \ln(R/L) \) both imply failure when \( M < 0 \), but lead to different values of \( \beta \) in first order approximations).
Reliability Index

Example 1: suppose that $M = R - L$
and $R$ and $L$ are normally distributed. Then

$$\mu_M = \mu_R - \mu_L$$

and

$$\sigma_M = \sqrt{\sigma_R^2 + \sigma_L^2}$$  \hspace{1cm} \text{(assuming independence)}$$

so that

$$\beta = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2}}$$

and

$$P[\text{failure}] = P[M < 0] = \Phi(-\beta)$$

This is exact, so long as $R$ and $L$ are normally distributed
and independent (if not independent then must involve
covariances in computation of $\sigma_M$).
Reliability Index

Example 2: suppose that $M = \ln \left( \frac{R}{L} \right) = \ln R - \ln L$ and $R$ and $L$ are lognormally distributed. Then

$$\mu_M = \mu_{\ln R} - \mu_{\ln L}$$

and

$$\sigma_M = \sqrt{\sigma^2_{\ln R} + \sigma^2_{\ln L}} \quad \text{(assuming independence)}$$

so that

$$\beta = \frac{\mu_{\ln R} - \mu_{\ln L}}{\sqrt{\sigma^2_{\ln R} + \sigma^2_{\ln L}}}$$

and

$$P[\text{failure}] = P[M < 0] = \Phi(-\beta)$$

This is exact, so long as $R$ and $L$ are lognormally distributed and independent (if not independent then must involve covariances in computation of $\sigma_M$).
Reliability Index

**Example 3:** suppose that \( M = \ln \left( \frac{R}{L} \right) = \ln R - \ln L \) and \( R \) and \( L \) are normally distributed. Then the distribution of \( M \) is complex and we must approximate its moments;

\[
\mu_M = \ln \mu_R - \ln \mu_L \quad \text{(to first order)}
\]

\[
\sigma_M \approx \sqrt{\left( \frac{\partial M}{\partial R} \right)_{\mu_R}^2 \sigma_{\ln R}^2 + \left( \frac{\partial M}{\partial L} \right)_{\mu_L}^2 \sigma_{\ln L}^2}
\]

\[
= \sqrt{\frac{\sigma_R^2}{\mu_R^2} + \frac{\sigma_L^2}{\mu_L^2}} = \sqrt{V_R^2 + V_L^2}
\]

now \( \beta = \frac{\ln \mu_R - \ln \mu_L}{\sqrt{V_R^2 + V_L^2}} \) \quad \text{It was} \quad \beta = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2}} \quad \text{in Example 1}

These are clearly different.
Hasover and Lind (1974) solved this ambiguity by mapping the set of system variables, $Z$, onto a set of standardized (mean zero, unit variance) and uncorrelated variables, $X$

$$X = A(Z - E[Z])$$

where the transformation matrix $A$ is the solution of

$$AC_ZA^T = I$$

where $C_Z$ is the matrix of covariances between the system variables, $Z$, and $I$ is the identity matrix. In terms of $Z$,

$$\beta = \min_{z \in L_Z} \sqrt{(z - E[Z])^T C_Z^{-1} (z - E[Z])}$$

where $L_Z$ is the failure surface. The value of $z$ which minimizes this is called the design point, $z^*$. 

Hasover-Lind’s reliability index is the minimum distance from the mean to the failure surface in standardized space (figure from Madsen, Krenk, and Lind, 1986)
Going Beyond Calibration

• Must move beyond calibration for real benefits of LRFD
• Simple probability-based methods take load and resistance distributions into account
  - nominal or characteristic resistance: \( R_n = k_R \mu_R \)
  - nominal or characteristic load: \( L_n = k_L \mu_L \)
• Design: \( \varphi R_n = \gamma L_n \)

\[
\text{P[failure]} = \text{P}[R < L] = \text{P}[R / L < 1]
\]
Going Beyond Calibration

- let $M = \ln(R/L)$
- then $P[\text{failure}] = P[M < 0]$

- $\beta$ is the reliability index
- typically $\beta$ ranges from 2.0 to 3.0
To determine both load and resistance factors:

We want to produce a design such that the mean and standard deviation of resistance satisfies

\[
P \left[ \frac{R}{L} < 1 \right] = P \left[ \ln R - \ln L < 0 \right] = \Phi(-\beta)
\]

In detail

\[
P[\ln R - \ln L < 0] = P \left[ Z < \frac{0 - \text{E}[\ln R - \ln L]}{\text{SD}[\ln R - \ln L]} \right] = \Phi \left( -\frac{\mu_{\ln R} - \mu_{\ln L}}{\sqrt{\sigma_{\ln R}^2 + \sigma_{\ln L}^2}} \right) = \Phi(-\beta)
\]

so that

\[
\frac{\mu_{\ln R} - \mu_{\ln L}}{\sqrt{\sigma_{\ln R}^2 + \sigma_{\ln L}^2}} = \beta \quad \Rightarrow \quad \mu_{\ln R} = \mu_{\ln L} + \beta \sqrt{\sigma_{\ln R}^2 + \sigma_{\ln L}^2} = \mu_{\ln L} + 0.75 \beta (\sigma_{\ln R} + \sigma_{\ln L})
\]

Now, since \( \mu_{\ln L} = \ln(\mu_L) - 0.5 \sigma_{\ln L}^2 \) and \( \mu_R = \exp(\mu_{\ln R} + 0.5 \sigma_{\ln R}^2) \) we get

\[
\mu_R = \mu_L \left[ \exp \left\{ 0.5 \sigma_{\ln R}^2 + 0.75 \beta \sigma_{\ln R} \right\} \exp \left\{ -0.5 \sigma_{\ln L}^2 + 0.75 \beta \sigma_{\ln R} \right\} \right]
\]

or

\[
\exp \left\{ 0.5 \sigma_{\ln R}^2 + 0.75 \beta \sigma_{\ln R} \right\} \mu_R = \exp \left\{ -0.5 \sigma_{\ln L}^2 + 0.75 \beta \sigma_{\ln R} \right\} \mu_L
\]
Writing this in terms of the nominal load and resistance
\[
R_n = k_R \mu_R \quad (k_R < 1)
\]
\[
L_n = k_L \mu_L \quad (k_L > 1)
\]
gives us
\[
\left[ \frac{\exp\left\{-0.5 \sigma_{\ln R}^2 - 0.75 \beta \sigma_{\ln R}\right\}}{k_R} \right] R_n = \left[ \frac{\exp\left\{-0.5 \sigma_{\ln L}^2 + 0.75 \beta \sigma_{\ln L}\right\}}{k_L} \right] L_n
\]
Recalling that our LRFD has the form \( \varphi R_n = \gamma L_n \) implies

Resistance factor: \( \varphi = \frac{\exp\left\{-0.5 \sigma_{\ln R}^2 - 0.75 \beta \sigma_{\ln R}\right\}}{k_R} \)

Load factor: \( \gamma = \frac{\exp\left\{-0.5 \sigma_{\ln L}^2 + 0.75 \beta \sigma_{\ln L}\right\}}{k_L} \)
If load factors are known (e.g. from structural codes) then the resistance factor becomes dependent on both the resistance variability and the load variability. In this case, our LRFD can be written

\[ \varphi R_n = \sum_i \gamma_i L_{n_i} \quad \Rightarrow \quad \varphi = \frac{\sum_i \gamma_i L_{n_i}}{R_n} = \frac{\sum_i \gamma_i L_{n_i}}{k_R \mu_R} \]

\[ = \frac{\sum_i \gamma_i Q_{n_i}}{k_R \mu_L} \left( \frac{\sqrt{1+V_L^2}}{\sqrt{1+V_R^2}} \right) \exp \left\{ -\beta \sqrt{\sigma_{lnR}^2 + \sigma_{lnL}^2} \right\} \]

where \( L_{n_i} = k_{L_i} \mu_{L_i} \)

\[ \mu_L = \sum_i \mu_{L_i} = \sum_i \frac{L_{n_i}}{k_{L_i}} \]

\[ V_L = \frac{\sigma_L}{\mu_L} = \sqrt{\sum_i \frac{V_{L_i}^2 \mu_{L_i}^2}{\mu_L}} \quad (\text{assuming loads are independent}) \]
Going Beyond Calibration

• thus, for given target reliability index and variances, the resistance factor can be computed as

\[
\varphi = \frac{\sum_{i} \gamma_{i} L_{ni}}{k_{R} \mu_{L}} \left( \frac{\sqrt{1 + V_{L}^2}}{\sqrt{1 + V_{R}^2}} \right) \exp \left\{ -\beta \sqrt{\sigma_{\ln R}^2 + \sigma_{\ln L}^2} \right\}
\]

which depends on
- coefficient of variation of load \( (V_{L}) \)
- coefficient of variation of resistance \( (V_{R}) \)
- load factors \( \gamma_{i} \)
- characteristic coefficients, \( k_{R} \) and \( k_{L_{i}} \)
Problems Implementing LRFD

- the coefficient of variation of resistance depends on;
  - variability in material properties
  - error in design models
  - measurement and correlation errors
  - construction variability

- with steel and concrete, the material property variability does not change significantly with location (quality controlled materials)

- with soils, the material property mean and variability change within a site and from site to site
Problems Implementing LRFD

- there is a dependence between resistance and load which is generally absent (or small) in structural engineering, e.g. shear strength is dependent on stress;

$$\tau_f = c + \sigma \tan \phi$$
Problems Implementing LRFD

• No common definition of “characteristic value”
  - often defined as a “cautious estimate of the mean”, but sometimes as a low percentile
  - we badly need a standard definition (median?)

• $V_R$ changes with intensity of site investigation
  - resistance factor should approach 1.0 as the site is more thoroughly investigated
  - this would lead to a complex table of resistance factors (however, see, e.g., AS 5100, AS 2159, AS4678, Eurocode 7, NCHRP507)
Future Directions

• probabilistic methods generally limited to “single random variable” models (e.g. $R$ vs. $L$)
• to consider the effect of spatial variability, random field simulation combined with finite element analysis is necessary (RFEM)
• the random field simulation allows the representation of a soil’s spatial variability
• the finite element analysis allows the soil to fail along “weakest paths” (decreased model error)
Conclusions

• geotechnical engineers led the way with Limit States concepts (1940’s) but have been slow to migrate to reliability-based design methods.
• the most advanced LRFD codes currently are AS 5100 and Eurocode 7.
• all current LRFD geotechnical codes have load and resistance factors calibrated from older WSD codes, with some adjustments based on engineering judgement and simple probability methods.